JOURNAL OF APPROXIMATION THEORY 41, 201–216 (1984)

Uniform Reciprocal Approximation Subject to Linear Constraints*

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Communicated by G. Meinardus

Received February 10, 1982

In this paper a general theory is presented for uniform approximation by reciprocals of elements of a linear subspace subject to linear constraints. In fact, the main results in the known theory of constrained linear approximation have analogues in this non-linear setting.

1. INTRODUCTION

In this paper we present a general theory for approximation by reciprocals of elements of a linear subspace subject to linear constraints. In fact, the main results of the known theory of constrained linear approximation have

* Research supported in part by the National Science and Engineering Research Council of Canada under Grant A8061 and by the National Science Foundation under Grant MCS-80-17056.

0021-9045/84 $3.00

analogues in this nonlinear setting. This setting is as follows. Let X be a (nonempty) compact subset of $[a, b]$ and denote by $C(X)$ the Banach space of all real-valued continuous functions defined on X normed with the uniform norm ($\|f\| = \max\{|f(x)| : x \in X\}$ for all $f \in C(X)$). Let V be an n-dimensional Haar subspace of $C(X)$ and let A be a compact set (of "restraint" linear functionals) in the dual, V^*, of V, such that for each p in V, $\alpha(p)$ is a continuous function on A. Set $R = \{1/p : p \in V, p(x) > 0 \forall x \in X, l(\alpha) \leqslant \alpha(p) \leqslant u(\alpha), \alpha \in A\}$ where l and u are extended real-valued functions on A with $l < +\infty$, $u > -\infty$, the set E_l (respectively E_u) on which l (respectively u) is finite is closed, l (respectively u) is continuous on E_l (respectively E_u) and $l(\alpha) \leqslant u(\alpha)$. We shall assume throughout that $R \neq \emptyset$. (Note that l (respectively u) is upper- (respectively lower-) semicontinuous on A.)

Let e_x represent point evaluation at x in X (i.e., $e_x(f) = f(x)$ for all $f \in C(X)$). In what follows, $f \in C(X)$ is called *admissible* provided $f(x) > 0$ for all $x \in X$ and if $\alpha = e_x$ for some $\alpha \in A$ and $x \in X$, then

$$\inf\left\{\left\|\frac{1}{f} - \frac{1}{p}\right\| : \frac{1}{p} \in R\right\} > \max\left\{\frac{1}{u(\alpha)} - \frac{1}{f(x)}, \frac{1}{f(x)} - \hat{l}(\alpha)\right\}$$

where $\hat{l}(\alpha) = 1/l(\alpha)$, if $l(\alpha) > 0$ and $+\infty$, otherwise. Note that we must necessarily have that $u(\alpha) > 0$ holds in this case since we have assumed $R \neq \emptyset$. Note that this inequality is assured if, for example, $l(\alpha) \leqslant f(x) \leqslant u(\alpha)$ for all $\alpha = e_x$ and $1/f \notin R$. (If $\alpha = e_x$ we will often write $l(x)$ and $u(x)$ for $l(e_x)$ and $u(e_x)$ from now on.) We are concerned then with approximating the reciprocals of such admissible f by elements of R. Thus, as usual, we say that $1/p^* \in R$ is a best approximation to $1/f$ on X, f admissible, provided

$$\left\|\frac{1}{f} - \frac{1}{p^*}\right\| = \inf_{1/p \in R}\left\|\frac{1}{f} - \frac{1}{p}\right\|.$$

Except for existence, the theory concerning characterization, uniqueness, etc., is analogous to the corresponding linear theory developed in [3–5]. The existence result is more complicated in this setting than in the linear theory where a straightforward appeal to compactness suffices. As in the reciprocal approximation theory without constraints the proof is more complicated when X is a compact subset of $[a, b]$ with isolated points [13, 14]. In addition, in the constrained setting we must add an additional assumption on the interplay between the constraints and interpolation properties of the approximants in order to prove an existence theorem.

In what follows we first develop the general theory and then illustrate it with some examples including monotone reciprocal approximation and bounded coefficient reciprocal approximation.

2. Main Results

We begin by proving two existence theorems. In both cases we must restrict the constraints allowed. Basically, we want the constraints to be such that either $1/p \in R$ implies there exists $M > 0$, independent of p, such that $\|p\| \leqslant M$ or in the event that this does not happen then we are able to control the values of certain elements of R on certain subsets of X. Our first theorem is for the case that $X = [a, b]$ and in this setting our assumption on the constraints is less restrictive. Thus, suppose that the constraints are such that corresponding to each $M > 0$ there exists $1/p \in R$ with $\|p\| > M$. Further, let $q \in V$ be any nonnegative element of V satisfying $\|q\| = 1$, $\alpha(q) \geqslant 0$ for all $\alpha \in E_l$, $\alpha(q) \leqslant 0$ for all $\alpha \in E_u$. Set $Z = \{x_i : q(x_i) = 0,\ x_i \in [a, b],\ i = 1,..., k\}$. Let $\{\lambda_i\}_{i=1}^k$ be a set of positive numbers satisfying $l(x_j) < \lambda_j < u(x_j)$ if $e_{x_j} \in A$ or $\lambda_j = u(x_j)$ if $e_{x_j} \in A$ with $l(x_j) = u(x_j)$. Then, assume that there exists $1/p \in R$ corresponding to q satisfying $p(x_i) \geqslant \lambda_i$, $i = 1,..., k$.

Theorem 1. *Let $f \in C[a, b]$, satisfying $f(x) > 0$ for all $x \in [a, b]$, be admissible. Further, suppose that the constraints are such that either $1/p \in R$ implies $\|p\| \leqslant M$, M independent of p, or the above assumption holds. Then there exists a best approximation in R to $1/f$.*

Proof. Let $\inf\{\|1/f - 1/p\| : 1/p \in R\} = \rho$. Let $\{1/p_k\}_{k=1}^{\infty} \subset R$ be such that $\|1/f - 1/p_k\| \downarrow \rho$. Now, if $\|p_k\|$ is a bounded sequence then the desired result follows by a straightforward compactness argument. Also, note that if $\rho = 0$, then since f is positive on $[a, b]$ it follows that for k sufficiently large, we will have $\{\|p_k\|\}$ is bounded in this case. Thus, we shall assume without loss of generality that $\rho > 0$ and $\|p_k\| \to \infty$. Define $q_k \in V$ by $q_k(x) = p_k(x)/\|p_k\|$ for all $x \in [a, b]$ and $k = 1, 2,...$. Since $\|q_k\| = 1$ for each k, we have by compactness the existence of $q \in V$ with $\|q\| = 1$, $q \geqslant 0$ for all $x \in [a, b]$ and where we assume without loss of generality that q_k converges uniformly to q on $[a, b]$. Let $Z = \{z_1,..., z_\nu\} \subset [a, b]$ be the complete set of zeros of q in $[a, b]$. By our Haar assumption we have that $\nu \leqslant n - 1$. (Observe that $Z = \phi$ is allowable.) Now for $x \in [a, b] \sim Z$ we have that $p_k(x) = \|p_k\|\, q_k(x) \to \infty$ as $k \to \infty$. Thus, for $x \in X \sim Z$, $1/f(x) \leqslant \rho$ and, hence, by continuity $1/f(x) \leqslant \rho$ for all $x \in [a, b]$. Thus, the proof will be completed if we can find $1/p \in R$ for which $p(x)$ is sufficiently large for all $x \in [a, b]$. In the case that $Z = \phi$, we have the existence of an $\eta > 0$ such that $q(x) \geqslant \eta$ for all $x \in [a, b]$. Thus, for k sufficiently large we have that $p_k(x) = \|p_k\|\, q_k(x) \geqslant \|p_k\|\, \eta/2 \geqslant 1/\rho$ implying that $\|1/f - 1/p_k\| \leqslant \rho$ so that $1/p_k$ is a desired best approximation. (In fact, $\|1/f - 1/p_k\| < \rho$ so $Z = \phi$ is impossible.) Thus, we shall assume that $Z \neq \phi$. Let $\alpha \in E_u$; then $u(\alpha) < \infty$ and for each k, $\alpha(p_k) \leqslant u(\alpha)$. Dividing both sides of this inequality by $\|p_k\|$ and letting

$k \to \infty$ gives that $\alpha(q) \leqslant 0$ for $\alpha \in E_u$. Likewise, $\alpha(q) \geqslant 0$ for $\alpha \in E_l$. Next, let $x_i \in Z$. If $e_{x_i} \notin A$ then select λ_i to satisfy $\lambda_i > 1/\rho$ so that $|1/f(x_i) - 1/\gamma| < \rho$ for any $\gamma \geqslant \lambda_i$. If $e_{x_i} \in A$ then select $\lambda_i = u(x_i)$ if $u(x_i) = l(x_i)$ and select $0 < \lambda_i < u(x_i)$ otherwise, so that $|1/f(x_i) - 1/\lambda_i| < \rho$, which can be done because of the admissibility of f. In addition, by the admissibility of f and the fact that $1/f \leqslant \rho$ and all numbers involved are positive, we have that for any γ, such that $\lambda_i \leqslant \gamma \leqslant u(x_i)$, $|1/f(x_i) - 1/\gamma| < \rho$. Now, invoking our assumption on the constraints, we have the existence of a $1/p \in R$ for which $p(x_i) \geqslant \lambda_i$, $i = 1,..., v$. Note that if $e_{x_i} \in A$ with $l(x_i) = u(x_i)$, then we necessarily have that $p(x_i) = \lambda_i$ in this case since $1/p \in R$. From our above discussion we have that $|1/f(x_i) - 1/p(x_i)| < \rho$ for $i = 1,..., v$. Thus, by continuity there exists an open set $\mathcal{O}$ such that $Z \subset \mathcal{O}$ and $|1/f(x) - 1/p(x)| < \rho$ for all $x \in \mathcal{O}$. Now, let B be a positive number: then $p(x) + Bq(x) \geqslant p(x)$ for all $x \in [a, b]$. Thus, since $1/f(x) \leqslant \rho$ and both $1/f(x) > 0$ and $1/p(x) > 0$ we must have that $|1/f(x) - 1/(p(x) + Bq(x))| < \rho$ for all $x \in \mathcal{O}$ and any $B > 0$. Since $[a, b] \sim \mathcal{O}$ is a compact set and q is positive on this set, we can now select B^* sufficiently large so that $|1/f(x) - 1/(p(x) + B^*q(x))| \leqslant \rho$ for all $x \in [a, b] \sim \mathcal{O}$. Thus, for this choice of B^* we have that $\|1/f - 1/(p + B^*q)\| \leqslant \rho$. To see that $1/(p + B^*q) \in R$ we simply note that $l(\alpha) \leqslant \alpha(p) \leqslant u(\alpha)$ for all $\alpha \in A$, $B^*\alpha(q) \geqslant 0$ if $\alpha \in E_l$ and $B^*\alpha(q) \leqslant 0$ if $\alpha \in E_u$, and $B^*\alpha(q)$ is always finite. (In fact, for B^* sufficiently large $\|1/f - 1/(p + B^*q)\| < \rho$, which is impossible. Thus, we can conclude further that, under the alternative assumption of Theorem 1, $\|p_k\| \to \infty$ is impossible.) Thus the argument is completed. ∎

In the section giving examples we shall show that both monotone reciprocal and bounded coefficient reciprocal approximation on intervals satisfy the hypotheses of this theorem.

For the case that X is a compact subset of $[a, b]$ we must invoke a slightly more stringent assumption to prove existence. Thus, if the side conditions are such that corresponding to each $M > 0$ there exists $1/p \in R$ such that $\|p\| > M$ then we assume the following: Let $q \in V$ be any nonnegative element of V satisfying $\|q\| = 1$, $\alpha(q) \geqslant 0$ for all $\alpha \in E_l$, $\alpha(q) \leqslant 0$ for all $\alpha \in E_u$. Set $Z = \{x_i \in X: q(x_i) = 0,\ i = 1,..., k\}$. Let $\{\lambda_i\}_{i=1}^k$ be a set of positive numbers satisfying $l(x_j) < \lambda_j < u(x_j)$ if $e_{x_j} \in A$ with $l(x_j) < u(x_j)$ and $\lambda_j = u(x_j)$ if $e_{x_j} \in A$ with $l(x_j) = u(x_j)$. Then assume that there exists $1/p \in R$ corresponding to q satisfying $p(x_i) = \lambda_i$ if x_i is an isolated point of X and $p(x_i) \geqslant \lambda_i$ otherwise, $i = 1,..., k$. With this assumption we can prove:

THEOREM 2. *Let $f \in C[a, b]$, satisfying $f(x) > 0$ for all $x \in X$, be admissible. Further, suppose that the constraints are such that either $1/p \in R$ implies $\|p\| \leqslant M$, M independent of p, or the above assumption holds. Then there exists a best approximation in R to $1/f$.*

Proof. The proof proceeds exactly as in the previous theorem except for the zeros of q that are isolated points of X. For these points we cannot conclude that $1/f \leqslant p$. Thus, at these points we need the assumption that $p(x_i) = \lambda_i$ holds where we select $\lambda_i = (\rho + 1/f(x_i))^{-1}$ if $e_{x_i} \notin A$, and choose λ_i as in the proof of Theorem 1 if $e_{x_i} \in A$. Once this additional restraint on p has been imposed, the remainder of the proof follows in the same manner. ∎

We now turn our attention to characterization and related questions. The following lemma demonstrates that the error function for reciprocal approximation $e(f, p) = |1/f - 1/p| = |(f-p)1/|f||p|$ is comonotone with the "standard" error function $d(f, p) = |f - p|$.

LEMMA 1. *Let $f > 0$ denote a fixed number and $p > 0$. Then $e(p) = |1/f - 1/p|$ and $d(p) = |f - p|$ increase (or decrease) simultaneously.*

Proof. If $0 < p < f$ then $e(p) = (f-p)/fp$ and $d(p) = f - p$; thus $e'(p) = -f^2/(fp)^2$ and $d'(p) = -1$ are both negative.

If $0 < f < p$ then $e(p) = (p - f)/fp$ and $d(p) = p - f$; thus $e'(p)$ and $d'(p)$ are both positive. ∎

Lemma 1 allows the development of characterization, uniqueness, and computation (Remes algorithm) for reciprocal approximation analogous to that developed in the case of the standard error function in [3–5]. (In fact, this theory can be developed for a wide class of "error measures" continuous and comonotone with $d(f, p)$ as indicated in Remark 4 of [3].) Following the development of [5] we shall say that if $l(\alpha) = u(\alpha)$ implies α is an isolated point of A, then equality condition 1 (EQC 1) is satisfied. We shall assume throughout this paper that EQC1 holds. In order to develop the desired characterization theory we must introduce the concepts of an extremal set and an augmented extremal set [5]. Thus, for $1/p \in R$ a set $S = I_1 \cup \{e_x\}_{x \in I_3} \subset V^*$ with $I_1 \subset A$ and $I_3 \subset X$ is called an *extremal set* for $1/f$ and $1/p$ provided

(i) $\alpha(p) = u(\alpha)$ (or $l(\alpha)$), $\alpha \in I_1$;

(ii) $|e_x(1/f - 1/p)| = \|1/f - 1/p\|$, $x \in I_3$; and

(iii) $e_x \notin I_1$ if $|e_x(1/f - 1/p)| = \|1/f - 1/p\|$.

To each $\alpha \in A$, we associate a set (possibly empty) of elements B_α in V^* such that if $1/q \in R$ then $\alpha(q) = l(\alpha)$ (or $u(\alpha)$) implies that, for each β in B_α, $\beta(q) = m(\beta)$ (or $n(\beta)$) where $m(\beta)$ (or $n(\beta)$) is some real number depending only on β. Then if $I_2 \subset \bigcup_{\alpha \in I_1} B_\alpha$, we say $S' = S \cup \{\beta\}_{\beta \in I_2}$ is an *augmented extremal set.*

An example of this is found in a combination of monotone and interpolating constraints. For example, if $R = \{1/p: p \in \Pi_5[0, 1], p'(x) \geqslant 0$ for all

$x \in [0, 1]$ and $p'''(\frac{1}{2}) = 0\}$ and S is an extremal set for some $1/f$ and $1/p$ that contains α where $\alpha(q) = q'(\frac{1}{2})$ (i.e., $p'(\frac{1}{2}) = 0$), then the two linear functionals β_1 and β_2, with $\beta_1(q) = q''(\frac{1}{2})$ and $\beta_2(q) = q^{(\text{iv})}(\frac{1}{2})$, adjoined to S will give an augmented extremal set for $1/f$ and $1/p$ with $m(\beta_1) = m(\beta_2) = 0$.

For $1/f$ and $1/p$ fixed, let $S^{\max}$ denote the maximal extremal set for $1/f$ and $1/p$, i.e., $S^{\max} = \{e_x : x \in X$ and $|e_x(1/f - 1/p)| = \|1/f - 1/p\|\} \cup \{\alpha \in A : \alpha(p) = l(\alpha)$ (or $u(\alpha)$) and if $\alpha = e_x$ for some $x \in X$ then $|e_x(1/f - 1/p)| < \|1/f - 1/p\|\}$. Further, let $S^{\max}_{\text{aug}}$ denote the maximal augmented extremal set for $1/f$ and $1/p$, i.e., $S^{\max}_{\text{aug}} = S^{\max} \cup \{\beta : \beta$ is associated with some $\alpha \in S^{\max}$ according to the previous definition$\}$. The elements of $S^{\max}$ are referred to as extreme points (of $1/f$ and $1/p$) and the elements of $S^{\max}_{\text{aug}} \sim S^{\max}$ are referred to as augmented extreme points. The cardinality of $S \subset S^{\max}_{\text{aug}}$ will be referred to as the order of S.

With these definitions it is now possible to utilize the concept of a generalized Haar space as introduced in [3]. This concept will be used in studying uniqueness (Theorems 6 and 7). Specifically, we say that V is *generalized Haar with respect to f and p*, where $1/p$ is a best approximation to $1/f$ from R with f admissible, provided that if $S^{\max}_{\text{aug}}$ for $1/f$ and $1/p$ has order t then $S^{\max}_{\text{aug}}$ contains $\min(t, n)$ elements which are linearly independent in V^*. V is said to be *generalized Haar* if V is generalized Haar for all pairs f and p with f admissible and $1/p$ a best approximation to $1/f$ from R.

Further, we say that V is *Haar* (on $\Omega = A \cup \{e_x\}_{x \in X}$) if any distinct n elements in Ω are linearly independent in V^*. Note that if V is Haar then V is generalized Haar, where $B_\alpha = \phi$, $\forall\, \alpha \in A$. Bounded coefficients approximation on $[0, 1]$ is an example where V is Haar. Monotone approximation (or more generally restricted derivatives approximation (R.D.A.)) is an example where V is generalized Haar but not Haar. (See the examples.)

We begin our study of the characterization of best approximation with the development of a Kolmogorov criterion. Let $1/f \in C(X) \sim R$ and $1/p^* \in R$. Define a "signature" function σ on $S^{\max}$, the maximal extremal set for $1/f$ and $1/p^*$, by

$$\begin{aligned}
\sigma(e_x) &= 1 && \text{if} \quad e_x\left(\frac{1}{f} - \frac{1}{p^*}\right) = \left\|\frac{1}{f} - \frac{1}{p^*}\right\| \\
\sigma(e_x) &= -1 && \text{if} \quad e_x\left(\frac{1}{f} - \frac{1}{p^*}\right) = -\left\|\frac{1}{f} - \frac{1}{p^*}\right\| \\
\sigma(\alpha) &= -1 && \text{if} \quad \alpha(p^*) = l(\alpha) \\
\sigma(\alpha) &= 1 && \text{if} \quad \alpha(p^*) = u(\alpha) \neq l(\alpha).
\end{aligned}$$

Note that σ has the opposite sign on the constraint extremals as compared with the corresponding linear theory. By our assumption that if $\alpha = e_x$ and both $|e_x(1/f - 1/p^*)| = \| 1/f - 1/p^* \|$ and $p^*(x) = \alpha(p^*) = l(\alpha) = l(x)$ or $p^*(x) = u(x)$ occur then we ignore the second condition, we have that σ is well defined (i.e., the value of $\sigma(\alpha)$ is determined by the first two equations in this case). In what follows, set $S_E = \{\alpha \in A: l(\alpha) = u(\alpha)\}$ and recall that this set consists of isolated points of A.

Finally we must assume

$$\exists \frac{1}{p_0} \in R \text{ such that } l(\alpha) < \alpha(p_0) < u(\alpha), \qquad \forall \alpha \in A \sim S_E. \tag{$*$}$$

Define $S^\sigma = \{\sigma(\gamma')\gamma': \gamma' \in S^{\max} \sim S_E\}$. Set $\bar{V} = \{p \in V: \alpha(p) = 0 \text{ for all } \alpha \in S_E\}$ and note that $\dim \bar{V} = n - \dim[S_E]$.

THEOREM 3 (KOLMOGOROV CRITERION). *Let $1/f \in C(X) \sim R$ and f be admissible. Then $1/p^* \in R$ is a best approximation to $1/f$ iff*

$$\max_{\gamma \in S^\sigma}(\gamma(p)) \geqslant 0 \qquad \text{for all } p \in \bar{V}.$$

Proof. $1/p^*$ is a best approximation to $1/f$ *iff* $\nexists p \in \bar{V}$ such that $p^* + \varepsilon p$ (for sufficiently small $\varepsilon > 0$) strictly improves upon p^* at the extrema $(S^{\max} \sim S_E)$ (consideration can be restricted to these extrema by the usual continuity and compactness argument and $(*)$ ensures that the improvement at $A \cap (S^{\max} \sim E)$ is strict without loss since if, for instance, $\alpha(p^* + \varepsilon p) = u(\alpha)$ then p can be replaced by $(1-\delta)p + \delta p_0$ for $\delta > 0$ sufficiently small), i.e., *iff* $\nexists p$ in $\bar{V}$ such that $\operatorname{sgn} \gamma'(p) = -\sigma(\gamma')$ for all $\gamma \in S^\sigma$, i.e., *iff* $\forall p$ in $\bar{V}$, $\max(\gamma(p)) \geqslant 0$. ∎

As a corollary of Theorem 3 we obtain the following very useful criterion for best approximation.

THEOREM 4 ("0 IN THE CONVEX HULL" CRITERION). *$1/p^*$ is a best approximation to $1/f \in C(X) \sim R$, f admissible, iff 0 is in the convex hull of some τ ($\leqslant \dim \bar{V} + 1$) elements of $S^\sigma|_{\bar{V}}$*, i.e., $0 = \sum_{i=1}^{\tau} \lambda_i \gamma_i$ on $\bar{V}$ where $\gamma_i \in S^\sigma$, $\lambda_i > 0$, $i = 1, \ldots, \tau$.

Proof. Let $\dim \bar{V} = m$ and identify $\bar{V}$ with $\mathbb{R}^m$. Then $\bar{V}^*$ can be identified with (another copy of) $\mathbb{R}^m$. Then $S^\sigma|_{\bar{V}} \subset \bar{V}^*$ and, for $\gamma \in S^\sigma|_{\bar{V}}$ and $p \in \bar{V}$, $\gamma(p)$ is realized as a "dot" product of two m-vectors. Thus $\max_{\gamma \in S^\sigma}(\gamma(p)) \geqslant 0$ for all $p \in \bar{V}$ represents the fact that for the set $S^\sigma|_{\bar{V}}$ there is no "direction" $p \in \bar{V}$ for which all vectors in $S^\sigma|_{\bar{V}}$ have a negative component. That is, $S^\sigma|_{\bar{V}}$ cannot lie in a half-space in $\mathbb{R}^m$; hence 0 must lie in the convex hull of (τ vectors in) $S^\sigma|_{\bar{V}}$. The fact that τ can be taken $\leqslant m + 1$ is Caratheodory's result. ∎

Observe that since we used the Kolmogorov criterion to obtain this result, we must still have the existence of $1/p_0 \in R$ such that $l(\alpha) < \alpha(p_0) < u(\alpha)$ for all α for which $l(\alpha) \neq u(\alpha)$. The characterization theorem proved in Theorem 4 is needed to developed a Remes-type computation theory analogous to that given in the linear theory [4].

THEOREM 5 (DE LA VALLÉE POUSSIN THEOREM). *Let $1/f \in C(X) \sim R$, f admissible, and $1/p \in R$. Let $\{e_x\}_{x \in I_1} \cup \{\alpha \in A : \alpha \neq e_x$ with $x \in I_1$, $\alpha(p) = l(\alpha)$ or $u(\alpha)$, and $l(\alpha) < u(\alpha)\}$ be a set of $k \leqslant \dim \bar{V} + 1$ linear functionals such that $0 \in$ convex hull of $[\{\sigma(e_x)\, e_x\} \cup \{\sigma(\alpha)\, \alpha\}]$ on $\bar{V}$ where $\sigma(e_x) = \operatorname{sgn}(1/f(x) - 1/p(x)) \neq 0$ $\forall x$. Then $\operatorname{dist}(1/f, R) \geqslant \min_{x \in I_1} |1/f(x) - 1/p(x)|$.*

Proof. Suppose $\sup_{x \in X} e(1/f, 1/p^*) < \min_{x \in I_1} e(1/f, 1/p)$ for some $1/p^* \in R$ where we may assume $l(\alpha) < \alpha(p^*) < u(\alpha)\ \forall\, \alpha \in A$ with $l(\alpha) \neq u(\alpha)$. Then for $x \in I_1$,

$$\begin{aligned}\operatorname{sgn}(p^*(x) - p(x)) &= \operatorname{sgn}\left(\frac{1}{p(x)} - \frac{1}{p^*(x)}\right)\\ &= \operatorname{sgn}\left[\left(\frac{1}{f(x)} - \frac{1}{p^*(x)}\right) - \left(\frac{1}{f(x)} - \frac{1}{p(x)}\right)\right]\\ &= -\operatorname{sgn}\left(\frac{1}{f(x)} - \frac{1}{p(x)}\right) = -\sigma(e_x).\end{aligned}$$

Furthermore, $\operatorname{sgn} \alpha(p^* - p) = -\sigma(\alpha)$. By assumption $\exists\, \lambda_i \geqslant 0$, such that

$$l = \sum_{i=1}^{\tau_1} \lambda_i \sigma(e_{x_i})\, e_{x_i} + \sum_{i=\tau_1+1}^{\tau} \lambda_i \sigma(\alpha_i)\, \alpha_i = 0 \qquad \text{on } \bar{V},$$

where not all λ_i $(i = 1, \ldots, \tau)$ are zero. But clearly $l(p^* - p) < 0$, and $p^* - p \in \bar{V}$, a contradiction. ∎

Next, we turn to the question of uniqueness. To answer this question we must prove a partial characterization result which differs from our previous characterization results as was done in [3] for the linear setting. In the following theorem, we assume $1/f \notin R$.

THEOREM 6. *If V is generalized Haar with respect to f and p ($1/p$ is a best approximation to $1/f$, f admissible, from R), then there exists an augmented extremal set for $1/f$ and $1/p$ of order $n + 1$.*

Proof. Let $\|1/f - 1/p\| = \max_X e(f, p) = \rho > 0$. Suppose there does not exist an augmented extremal set of order $n + 1$. Let $S_{\text{aug}}^{\max} = \{\alpha_\nu\}_{\nu=1}^{r} \cup \{\beta_\mu\}_{\mu=r+1}^{s} \cup \{e_{x_i}\}_{i=s+1}^{t}$ where $t \leqslant n$ and if α_ν, some ν, $1 \leqslant \nu \leqslant r$, has the form e_x then $|e_x(1/f - 1/p)| < \rho$; whereas it is possible for e_{x_i}, some i,

$s+1 \leqslant i \leqslant t$, that e_{x_i} is both an error extremal (i.e., $|e_{x_i}(1/f - 1/p)| = \rho$) and a constraint extremal (i.e., $e_{x_i}(p) = l(x_i)$ or $u(x_i)$). Recall that $l(\alpha) < +\infty$, $u(\alpha) > -\infty$, $l(\alpha)$ is upper semicontinuous, $u(\alpha)$ is lower semicontinuous, and $l(\alpha) < u(\alpha)$ except possibly at isolated points of A.

Next, because of the independence of the functionals in $S_{\text{aug}}^{\max}$ by our generalized Haar assumption, there exists a nonzero element q of V such that

$$\alpha_\nu(q) = \begin{cases} 1, & l(\alpha_\nu) < u(\alpha_\nu) \text{ and } \alpha_\nu(p) = l(\alpha_\nu) \\ -1, & l(\alpha_\nu) < u(\alpha_\nu) \text{ and } \alpha_\nu(p) = u(\alpha_\nu) \\ 0, & l(\alpha_\nu) = u(\alpha_\nu), \end{cases}$$

$\nu = 1,\ldots, r$, and $q(x_i) = \operatorname{sgn}(f(x_i) - p(x_i)) = -\operatorname{sgn}(1/f(x_i) - 1/p(x_i))$, $i = s+1,\ldots, t$. Then, for sufficiently small $\varepsilon > 0$, we have by Lemma 1, since $|f(x_i) - (p+\varepsilon q)(x_i)| = |(f-p)(x_i) - \varepsilon q(x_i)| < |(f-p)(x_i)|$, that $e(f(x_i), (p+\varepsilon q)(x_i)) < \rho$, $i = s+1,\ldots, t$. But q and $f - p$ are continuous on X and therefore $\operatorname{sgn} q(x) = \operatorname{sgn}(f-p)(x)$ in a neighborhood $\mathcal{N}$ of $\{x_{s+1},\ldots, x_t\}$ in X. Thus, also $|f(x) - (p+\varepsilon q)(x)| < |(f-p)(x)|$ and therefore by Lemma 1, $e(f(x), (p+\varepsilon q)(x)) < \rho$ in $\mathcal{N}$ for sufficiently small $\varepsilon > 0$. Furthermore, $e(f(x), p(x)) < \rho$ in the compact set $X - \mathcal{N}$. Thus, since $e(\cdot, \cdot)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^+$, for sufficiently small $\varepsilon > 0$, $e(f(x), (p+\varepsilon q)(x)) < \rho$ in $(X - \mathcal{N}) \cup \mathcal{N}$ and so $\| 1/f - 1/(p+\varepsilon q)\| = \max_X e(f(x), (p+\varepsilon q)(x)) < \rho$.

Now we must check that $1/(p+\varepsilon q)$ belongs to R for $\varepsilon > 0$ small enough and then $1/(p+\varepsilon q)$ will furnish a better approximation to $1/f$ than $1/p$, yielding the desired contradiction. Now, for $\nu = 1,\ldots, r$ and sufficiently small $\varepsilon > 0$, it is easily seen that $l(\alpha_\nu) \leqslant \alpha_\nu(p+\varepsilon q) = \alpha_\nu(p) + \varepsilon\alpha_\nu(q) \leqslant u(\alpha_\nu)$ with strict equality wherever $l(\alpha_\nu) \neq u(\alpha_\nu)$. Next, suppose for some i, $s+1 \leqslant i \leqslant t$, that the error extremal e_{x_i} is also a constraint extremal. Assume that $e_{x_i}(1/f - 1/p) = -\rho$ so that e_{x_i} is a negative error extremal, $q(x_i) = 1$ and $(p+\varepsilon q)(x_i) > p(x_i)$. Now, if $e_{x_i}(p) = u(x_i)$ then

$$\frac{1}{u(x_i)} - \frac{1}{f(x_i)} = \frac{1}{p(x_i)} - \frac{1}{f(x_i)} = \left\| \frac{1}{f} - \frac{1}{p} \right\|$$

which violates the admissibility assumption for f. Thus, we must have that $l(x_i) < u(x_i)$ and $e_{x_i}(p) = l(x_i)$ in this case implying that $l(x_i) > 0$ and $l(x_i) < e_{x_i}(p+\varepsilon q) < u(x_i)$ for $\varepsilon > 0$ sufficiently small. A similar argument holds when $e_{x_i}(1/f - 1/p) = \rho$. Now, set

$$A' = (\{\alpha_\nu\}_{\nu=1}^r \cap \{\alpha \in A : l(\alpha) \neq u(\alpha)\}) \\ \cup \{e_x \in \{e_{x_i}\}_{i=s+1}^t : e_x(p) = l(x) \text{ or } u(x)\}.$$

Then for sufficiently small $\varepsilon > 0$, we have $l(\alpha) < \alpha(p+\varepsilon q) < u(\alpha)$ for

$\alpha \in A'$. But by our assumptions, $\alpha(p)$ and $\alpha(q)$ are continuous on A. Thus also $l(\alpha) < \alpha(p + \varepsilon q) < u(\alpha)$ in a neighborhood $\mathscr{A}'$ of A' in A for sufficiently small $\varepsilon > 0$. Further, we have $l(\alpha) < \alpha(p) < u(\alpha)$ for $\alpha \in A \sim (\mathscr{A}'' \cup \mathscr{A}')$, where $\mathscr{A}'' = \{\alpha \in A : l(\alpha) = u(\alpha)\}$. By assumption, $\mathscr{A}''$ consists of just isolated points of A; thus $\mathscr{A}''$ is open in A and so $A \sim (\mathscr{A}'' \cup \mathscr{A}')$ is compact in A. Thus by lower semicontinuity $\alpha(p) - l(\alpha)$ and $u(\alpha) - \alpha(p)$ achieve positive minima on $A \sim (\mathscr{A}'' \cup \mathscr{A}')$ and so for ε sufficiently small $l(\alpha) < \alpha(p + \varepsilon q) < u(\alpha)$ on this set ($\alpha(q)$ is continuous and therefore bounded on $A \sim (\mathscr{A}'' \cup \mathscr{A}')$). Noting finally that, for α in $\mathscr{A}''$, $\alpha(q) = 0$, we conclude that $l(\alpha) \leqslant \alpha(p + \varepsilon q) \leqslant u(\alpha)$, for all α in A, for sufficiently small $\varepsilon > 0$. ∎

LEMMA 2. *For* $0 < \lambda < 1$, $f > 0$ *fixed*, $p > 0$, $p^* > 0$, $|e(f, \lambda p + (1 - \lambda) p^*)| \leqslant \max(|e(f, p)|, |e(f, p^*)|)$, *with equality holding only if* $p = p^*$.

Proof. This follows from Lemma 1 since $|f - \lambda p - (1 - \lambda) p^*| \leqslant \lambda |f - p| + (1 - \lambda)|f - p^*| \leqslant \max(|f - p|, |f - p^*|)$ with equality holding only if $|f - p| = |f - p^*|$. But if $f - p = -(f - p^*)$ then f and $\lambda p + (1 - \lambda) p^*$ lie strictly between p and p^*, so the inequality is strict. ∎

LEMMA 3. *The set of best approximations in R is reciprocally convex (i.e., if $1/p$ and $1/p^*$ are best approximations to $1/f$, then $1/(\lambda p + (1 - \lambda) p^*)$ is a best approximation to $1/f$ for $0 < \lambda < 1$).*

Proof. By Lemma 2, $\|e(f, \lambda p + (1 - \lambda) p^*)\| \leqslant \max(\|e(f, p)\|, \|e(f, p^*)\|)$ and so $1/(\lambda p + (1 - \lambda) p^*)$ is a best approximation in R to $1/f$. ($\lambda p + (1 - \lambda) p^*$ satisfies the linear restraints since p and p^* do). ∎

THEOREM 7. *If V is generalized Haar with respect to f and p, with f admissible, then $1/p$ is the unique best approximation in R to $1/f$.*

Proof. First observe that if $\text{dist}(1/f, R) = 0$ then $1/f$ is its own unique best approximation from R. Thus, assume $\text{dist}(1/f, R) > 0$. Now suppose $1/p$ and $1/p^*$ are best approximations in R to $1/f$. Then by Lemma 3 so is $1/p^{**} = 1/(\frac{1}{2} p + \frac{1}{2} p^*)$. Thus, by Theorem 6 there exists a maximal augmented extremal set S for $1/f$ and $1/p^{**}$ of order $t \geqslant n + 1$, say, $S = \{\alpha_\nu\}_{\nu=1}^r \cup \{\beta_\mu\}_{\mu=r+1}^s \cup \{e_{x_i}\}_{i=s+1}^t$. Then $l(\alpha_\nu) \leqslant \alpha_\nu(p^{**})$, $\alpha_\nu(p)$, $\alpha_\nu(p^*) \leqslant u(\alpha_\nu)$ and $\alpha_\nu(p^{**}) = l(\alpha_\nu)$ (or $u(\alpha_\nu)$) therefore implies that $\alpha_\nu(p) = \alpha_\nu(p^*) = l(\alpha_\nu)$ (or $u(\alpha_\nu)$), $\nu = 1,..., r$. As a result, $\beta_\mu(p) = \beta_\mu(p^*) = m(\beta_\mu)$ (or $n(\beta_\mu)$), $\mu = r + 1,..., s$. Finally, $|e(f(x_i), p^{**}(x_i))| = \|e(f, p)\| = \|e(f, p^*)\|$ implies by Lemma 2 that $p(x_i) = p^*(x_i)$ since $\rho = |e(f(x_i), p^{**}(x_i))| \leqslant \max(|e(f(x_i), p(x_i))|, |e(f(x_i), p^*(x_i))|) \leqslant \rho$, $i = s + 1,..., t$. But since $t \geqslant n + 1$ and some n of the elements of S are linearly independent in the dual of V, we have $p = p^*$. ∎

COROLLARY 1. *If V is generalized Haar, then any best approximation $1/p$ in R to $1/f$, f admissible, is unique.*

A complete strong uniqueness theory also exists for this problem [6]. It turns out that the order of strong uniqueness is dependent upon the set of constraints. Thus, for example, in $R = \{1/p; p \in \Pi_n, p(x) > 0 \text{ and } p'(x) \geqslant 0 \ \forall x \in [0, 1]\} \subset C[0, 1]$ strong uniqueness holds with order $\frac{1}{2}$. That is, if $f \in C[0, 1], f > 0$ and $1/p_f \in R$ is the unique best approximation to $1/f$ from R then given $M > 0$ there exists $\gamma = \gamma(f, M) > 0$ such that

$$\left\| \frac{1}{f} - \frac{1}{p} \right\| \geqslant \left\| \frac{1}{f} - \frac{1}{p_f} \right\| + \gamma \left\| \frac{1}{p} - \frac{1}{p_f} \right\|^2$$

for all $1/p \in R$ satisfying $\|1/p\| \leqslant M$. Note that the power of $\|1/p - 1/p_f\|$ is the reciprocal of the order. Thus, for this problem a Lipschitz continuity of the best approximation operator of order $\frac{1}{2}$ holds. That is,

$$\left\| \frac{1}{p_f} - \frac{1}{p_g} \right\| \leqslant \beta \left\| \frac{1}{f} - \frac{1}{g} \right\|^{1/2}.$$

See [6] for complete details.

Following a development analogous to that in [4], we could now demonstrate a Remes-like algorithm which converges for constrained reciprocal approximation provided V is Haar. Because, however, of the success of the differential correction algorithm in rational approximation [1, 7–9, 11, 12], we will consider an adaptation of this latter algorithm to our setting.

Let $V = \langle \phi_1, ..., \phi_n \rangle$ and for $p \in V$ write $p = \sum_{j=1}^{n} p_j \phi_j$. In this setting the differential correction algorithm is as follows, where we now assume X is a finite set.

ALGORITHM (RESTRICTED RECIPROCAL DIFFERENTIAL CORRECTION (RRDC)).

(i) *Choose* $1/p_0 \in R$.

(ii) Having found $1/p_k \in R$ with $\|1/f - 1/p_k\| = \Delta_k$, *choose* p_{k+1} *as a solution to the problem: Find* $p = \sum_{j=1}^{n} p_j \phi_j \in V$ *which solves*

$$\textit{Minimize}: \max_{x \in X} \frac{|(1/f(x)) p(x) - 1| - \Delta_k p(x)}{p_k(x)},$$

$$\textit{Subject to}: l(\alpha) \leqslant \alpha(p) \leqslant u(\alpha), \alpha \in A,$$

$$\textit{and} \ |p_j| \leqslant K, j = 1, ..., n.$$

(iii) *Continue until some stopping criterion is met. Here K is some large positive number; the constraints involving K are present to ensure the problem in* (ii) *has a solution. Note that the linear functional constraints are just linear constraints on the coefficients of p; that is, since* $\{\phi_1, \ldots, \phi_n\}$ *is a basis for V, then* $l(\alpha) \leqslant \alpha(p) \leqslant u(\alpha)$ *is equivalent to* $l(\alpha) \leqslant \alpha(p_1\phi_1 + \cdots + p_n\phi_n) \leqslant u(\alpha)$, *i.e.,* $l(\alpha) \leqslant p_1\alpha(\phi_1) + \cdots + p_n\alpha(\phi_n) \leqslant u(\alpha)$.

One common stopping criterion is to stop when $(\Delta_k - \Delta_{k+1})/\Delta_k < \varepsilon$ for some prescribed $\varepsilon > 0$, selecting $1/p_{k+1}$ as the approximation returned by the algorithm if $\Delta_{k+1} < \Delta_k$, and selecting $1/p_k$ otherwise. A convenient way of choosing $1/p_0$ is to minimize $\max_{x \in X} |(1/f(x))\, p(x) - 1|$ subject to the constraints in (ii) above along with additional constraints to force $p(x) \geqslant \varepsilon_1$ for some $\varepsilon_1 > 0\ \forall\, x \in X$.

The following theorem can be proved by arguments similar to but somewhat simpler than those in [1] and [9].

THEOREM 8. *Suppose a best approximation* $1/p^* \in R$ *exists for* $1/f \in C(X) \sim R$, *f admissible, and satisfies* $|p_j^*| \leqslant K$, $j = 1, \ldots, n$. *Then*:

(i) *The RRDC algorithm converges monotonically and at least linearly.*

(ii) *If X contains at least* $n+1$ *distinct points and either* $S^{\max} \cap A = \phi$ *or* 0 *is in the* interior *of the convex hull of* $S^{\sigma}|_{\Gamma}$ *(see Theorem* 4), *then the converge is quadratic.*

We observe that the key to the proof in part (ii) is that strong uniqueness holds under the assumptions of (ii). Although finiteness of X and A are not required for the theorem, they are required to run the algorithm in the usual way (i.e., solving (ii) by linear programming).

EXAMPLE. Let $1/f(x) = e^{-x}$, $X = \{0, 0.1, 0.2, \ldots, 10\}$,

$$R = \left\{ \frac{1}{p} = \frac{1}{p_1 + p_2 x + p_3 x^2} : p_1 + p_2 = 1,\ p(10) \geqslant 1000 \text{ and } p(x) > 0\ \forall\, x \in X \right\}.$$

Applying the RRDC algorithm with $K = 100$ on a CDC Cyber 172 computer (which has roughly 15 digits of accuracy) and a modified version of the code in [8], after five iterations and 2.2 seconds execution time we obtained

$$\frac{1}{p^*(x)} = \frac{1}{0.75409 + 0.24591x + 9.96787x^2}$$

with error norm 0.32610 and

$$\frac{1}{f(0)}-\frac{1}{p^*(0)}=-0.32610, \qquad \frac{1}{f(0.6)}-\frac{1}{p^*(0.6)}=0.32610.$$

In our notation we have $A=\{\alpha_1,\alpha_2\}$ where $\alpha_1(p)=p_1+p_2$, $\alpha_2(p)=p(10)$, $l(\alpha_1)=u(\alpha_1)=1$, $l(\alpha_2)=1000$, $u(\alpha_2)=+\infty$, $\bar{V}=\{p\in V=\Pi_2: p_1+p_2=0\}=\{p_1-p_1x+p_3x^2\}=\{p_1(1-x)+p_3x^2\}$ and $S_E=\{\alpha_1\}$. The computed result was $S^{\max}=\{e_0,e_{0.6},\alpha_1,\alpha_2\}$, with $\sigma(e_0)=-1$, $\sigma(e_{0.6})=1$, $\sigma(\alpha_2)=-1$, so $S^\sigma=\{-e_0,e_{0.6},-\alpha_2\}$. To show $1/p^*$ is a best approximation, we wish to show $0\in$ convex hull of $S^\sigma|_{\bar{V}}$, that is, $\exists$ nonnegative λ_1, λ_2, λ_3 (not all zero) with $-\lambda_1e_0(p)+\lambda_2e_{0.6}(p)-\lambda_3\alpha_2(p)=0\ \forall p\in\bar{V}$. This is equivalent to showing $\exists\,\lambda_1\geqslant 0$, $\lambda_2\geqslant 0$, $\lambda_3\geqslant 0$ with $\lambda_1e_0(1-x)-\lambda_2e_{0.6}(1-x)+\lambda_3\alpha_2(1-x)=0$, $\lambda_1e_0(x^2)-\lambda_2e_{0.6}(x^2)+\lambda_3\alpha_2(x^2)=0$, $\lambda_1+\lambda_2+\lambda_3=1$. Solving, we get $\lambda_1=4.324/14.36$, $\lambda_2=10/14.36$, $\lambda_3=0.036/14.36$. In fact, it is easily seen that $0\in$ the interior of the convex hull of $S^\sigma|_{\bar{V}}$; thus $1/p^*$ is (strongly) unique, and the convergence of the RRDC is quadratic.

A pair of constraints like $p_1+p_2\leqslant 1$ and $p_1+p_2\geqslant 1$ could cause difficulty in practice because round-off error could make them appear inconsistent, thus terminating the program. If this happens, one can eliminate these constraints, and with a little reprogramming replace R by

$$\left\{\frac{1}{p}=\frac{1}{x+p_1(1-x)+p_3x^2}: p(10)\geqslant 1000, p(x)>0\ \forall\, x\in X\right\}.$$

3. Examples

In this section we discuss, for purposes of illustration, the two examples of monotone reciprocal approximation and bounded coefficient reciprocal approximation (or more generally V Haar (on Ω)).

Monotone reciprocal approximation [2]. Let $(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_q)$ be a $(q+1)$-tuple with elements equal to ± 1 and let $\{k_i\}_{i=0}^q$ be a fixed set of $q+1$ integers satisfying $0<k_0<k_1<\cdots<k_q\leqslant n-1$. Then $R=\{1/p: p\in\Pi_{n-1}[a,b],\ p(x)>0$ and $\varepsilon_i p^{(k_i)}(x)\geqslant 0\ \forall x\in[a,b],\ i=0,1,\ldots,q\}$. Thus $A=\{\varepsilon_i e_x^{k_i}: x\in[a,b]$ and $i\in\{0,1,\ldots,q\}\}$ is the set of restraining functionals with $l(\alpha)=0$ and $u(\alpha)=\infty$ for all $\alpha\in A$. (Here e_x^k denotes point evaluation of the kth derivative at x, i.e., $e_x^k(f)=f^{(k)}(x)$.) For the fact that $V=\Pi_{n-1}[0,1]$ is generalized Haar see Theorem 4 and Remark 4 of [3]. (The proof of Theorem 4 of [3] is modified in our present case by replacing the phrase "$p_1(x)$ is clearly a best approximation to $f(x)-p_2(x)$" by the phrase "$p_1(x)$ is clearly a best weighted approximation to $f(x)-p_2(x)$ with weight $w(x) = 1/(f(x)\,p(x))$" (observe that the "0 in the convex hull" criterion holds for $f - p_2$ approximated by p_1 with weight $w(x)$ using the t extremals of the first $r+1$ columns of $\mathscr{E}$ implying p_1 is best by use of Theorem 4 of this present paper applied to $1/f(x)$ and $1/p(x)$).)

To see that the assumption in the hypothesis of Theorem 1 holds, note first that $\nexists\, e_x \in A$. Thus, given any nonzero $\tilde{q} \in \Pi_{n-1}$ with zeros $Z = \{x_1, ..., x_k\}$, (necessarily) $k \leqslant n-1$, and $\{\lambda_i\}_{i=1}^k$ positive, we can find $1/p \in R$ such that $p(x_i) \geqslant \lambda_i$, $i = 1, ..., k$—namely, $1/p \equiv 1/\lambda$ where $\lambda = \max_{1 \leqslant i \leqslant k} \lambda_i + 1$. We conclude in particular, from the above theory, that there exists a unique best monotone reciprocal approximation $1/p$ to $1/f$.

Bounded coefficients reciprocal approximation. Let $\{k_i\}_{i=0}^q$ be a fixed set of $q+1$ integers satisfying $0 \leqslant k_0 < k_1 < \cdots < k_q \leqslant n-1$. Set $R = \{1/p : p \in \Pi_{n-1}(X),\ l_i \leqslant p^{(k_i)}(0) \leqslant u_i,\ i = 0, 1, ..., q\}$, $X \subset [a, b]$, where either $a \geqslant 0$ or $b \leqslant 0$ and X is compact. Thus, $A = \{e_0^{k_i}\}_{i=0}^q$ with $l(\alpha) = l_i$ and $u(\alpha) = u_i$ for $\alpha = e_0^{k_i} \in A$. For the fact that $V = \Pi_{n-1}(X)$ is Haar (on Ω) see [3]. To see that the assumptions in the hypotheses of Theorems 1 and 2 hold, consider $\tilde{q} \in \Pi_{n-1}$ where $\|\tilde{q}\| = 1$, $\tilde{q} \geqslant 0$, $\tilde{q}^{(k_i)}(0) \geqslant 0$ if $l_i > -\infty$ and $\tilde{q}^{(k_i)}(0) \leqslant 0$ if $u_i < \infty$, $i = 0, 1, ..., q$. Let $Z_1 = \{k_i : \tilde{q}^{(k_i)}(0) = 0\}$ have order μ, and let $Z = \{x_i\}_{i=1}^k$ be the zeros of $\tilde{q}$ in $[a, b]$. Note that since V is Haar (on Ω), $\{e_0^{k_i} : k_i \in Z_1\} \cup \{e_{x_i}\}_{i=1}^k$ is a linearly independent set in V^* where if e_0 happens to occur in both sets, it is not counted twice (q is a nonzero element of V vanishing under all these functionals and there are therefore fewer than n members in this set of functionals). Let $\{\lambda_i\}_{i=1}^k$ be a set of positive numbers satisfying the requirement that if $k_0 = 0$ and $x_i = 0$ (for some i) then $l_0 < \lambda_i < u_0$ or $\lambda_i = u_0$ if $u_0 = l_0$. Thus we can find $p \in V$ such that $p(x_i) = \lambda_i$, $i = 1, ..., k$, and $l_i \leqslant p^{(k_i)}(0) \leqslant u_i$, $\forall\, k_i \in Z_1$. Consider now $p_B = p + B\tilde{q}$, $B > 0$. It is easy to verify that for B large enough p_B satisfies all the constraints and is positive and therefore $1/p_B$ is in R. Thus for B sufficiently large p_B satisfies the assumptions in the hypotheses of Theorems 1 and 2 (corresponding to q). We conclude in particular, from the above theory, that there exists a unique best bounded coefficients reciprocal approximation $1/p$ to $1/f$.

Note that the discussion in the preceding example depends only on the fact that V is Haar (on Ω). Hence we have the following result.

THEOREM 9. *Let V be Haar (on Ω). Then there exists a unique best reciprocal approximation $1/p$ to $1/f$.*

Remark 1. If V is Haar (on Ω), then $\tau = \dim \bar{V} + 1$ in Theorem 4 (the "0 in the convex hull" criterion for best approximation) and the "exchange procedure" in the Remes algorithm (which we do not develop in this paper) follows immediately from knowledge of the sign pattern $(\sigma(\gamma_i'),\ i = 1, ..., \tau)$ in $0 = \sum_{i=1}^{\tau} \lambda_i \sigma(\gamma_i')\, \gamma_i'$, $\lambda_i > 0$. The determination of the sign pattern is an algebraic problem and has been worked out completely [4] in the case of bounded coefficients approximation (in fact, for approximation with general "pyramid" (see [4] for definition) constraints). If, for example, $a=0$ and $\bar{V}=V$ (whence $\tau=n+1$) and $\gamma_1', \gamma_2', ..., \gamma_{n+1}' = e_0^{k_q}, e_0^{k_{q-1}}, ..., e_0^{k_0}, e_{x_1}, e_{x_2}, ..., e_{x_{n-q}}$, where $x_1 < x_2 < \cdots < x_{n-q}$, then

$$\sigma(\gamma_i') \cdot \sigma(\gamma_{i+1}') = \begin{cases} (-1)^{k_{q-i-1}-k_{q-i+1}}, & i=1,2,...,q+1\ (k_{-1}=0) \\ -1, & i=q+2,...,n. \end{cases}$$

If instead $b=0$ and $\gamma_1', \gamma_2', ..., \gamma_{n+1}' = e_{x_1}, e_{x_2}, ..., e_{x_{n-q}}, e_0^{k_0}, e_0^{k_1}, ..., e_0^{k_q}$, then

$$\sigma(\gamma_i') \cdot \sigma(\gamma_{i+1}') = -1, \qquad i=1,2,...,n.$$

If $\bar{V} \subsetneqq V$ (i.e., $\tau < n+1$) then the sign pattern also follows in this case from the above in an obvious way (i.e., the equality constraint functionals are reinserted and the above formulas are applied).

Remark 2. For further examples where V is Haar (on Ω) or V is generalized Haar, see, e.g., [3–5].

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